Optimization Letters
https://doi.org/10.1007/s11590-020-01604-0

ORIGINAL PAPER

Stability analysis for set-valued equilibrium problems with applications to Browder variational inclusions

L. Q. Anh[1] · P. T. Duoc[2,3] · T. N. Tam[4,5] · N. C. Thang[6]

Received: 11 May 2019 / Accepted: 26 May 2020

Abstract
In this paper we study two types of strong set-valued equilibrium problems in Hausdorff locally convex topological vector spaces. Under suitable assumptions, stability in the sense of Hausdorff continuity of solutions is established. Main results are applied to Browder variational inclusions.

Keywords Stability analysis · Set-valued equilibrium problem · Hausdorff continuity · Browder variational inclusion

1 Introduction

In recent years, set-valued equilibrium problems have been received much attention of authors since they not only generalize the single-valued equilibrium problems, but also serve as unified models to investigate set-valued variational inequalities. It

✉ T. N. Tam
tranngoctam@tdtu.edu.vn

L. Q. Anh
quocanh@ctu.edu.vn

P. T. Duoc
ptduoc@vttu.edu.vn

N. C. Thang
nguyenchithang@gmail.com

1 Department of Mathematics, Teacher College, Can Tho University, Can Tho, Vietnam

2 University of Science, Ho Chi Minh City, Vietnam

3 Vietnam National University, Ho Chi Minh City, Vietnam

4 Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam

5 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

6 Department of Mathematics, Nam Can Tho University, Can Tho, Vietnam

Published online: 03 June 2020

is well-known that variational inequality includes Browder variational inclusion, a generalization of Browder–Hartman–Stampacchia variational inequality, as a special case. The Browder-Hartman-Stampacchia variational inequality has a wide application, especially to the surjectivity of set-valued maps and to nonlinear elliptic boundary value problems, see e.g., [1,2]. Also, the equilibrium problem has many applications to different areas such as physis, economics, engineering, transportation, chemistry, biology, etc (see [3]).

Existence of solutions to set-valued equilibrium problems has been studied by several researchers in the literature [4–8]. The stability of solutions for these problems is also a motivation for many authors. In [9–12], the authors obtain the (semi)continuity of the solution maps to set-valued equilibrium problems by using a linear scalarization method which is an effective tool for studying set-valued equilibrium problems in the weak types. Besides, many results on the stability in the sense of Hölder/Lipschitz continuity of the solution maps to set-valued equilibrium problems are archived, see [13,14].

Noting that when investigating practical problems, because of measure devices or statistical results, obtained data are approximate, the models of these problems are also approximations. This means that exact solutions to such practical problems are, in fact, also approximate ones. Also, most numerical methods used to solve mathematical models often produce approximate solutions to the models, and hence in computing and solving practical problems usually obtains approximate ones.

Therefore, studying approximate solutions to mathematical models plays important roles in both theoretical and computational aspects, and thus it has motivated and inspired many researchers to investigate (see [15–23]). However, as far as we know, there have been only few works devoted to upper and lower semicontinuity property of approximate solutions, which have close relations with well-posedness property, an important topic, for the reference problems (see [16,20–22]). In the other word, the obtained results related to the semicontinuities of approximate solution sets have not been described important roles of them.

From above observations, we aim to study the stability conditions for approximate solutions to two types of strong set-valued equilibrium problems. More precisely, sufficient conditions for the approximate solution maps to be continuous in the sense of Hausdorff are established. Our main results are new and include the previous ones in the literature [18,23]. As an application, the main results are applied to the Browder variational inclusions.

The outline of the remainder of the paper is organized as follows. Section 2 recalls some definitions and their properties needed in what follows. We introduce, in Sect. 3, two types of strong set-valued equilibrium problems and establish sufficient conditions for the Hausdorff continuity of the solution maps to such problems. In the last section, Sect. 4, we discuss the Hausdorff continuity of the solution maps to the Browder variational inclusions.

2 Preliminaries

Throughout the paper, unless otherwise stated, let X be a Hausdorff locally convex topological vector space, Y be a real topological vector space and K be a nonempty subset of X. Assume that C is a pointed closed convex cone in Y with nonempty interior ($\mathrm{int}C \neq \emptyset$) and $e \in \mathrm{int}C$ is given.

Let $F : K \times K \rightrightarrows Y$ is a set-valued map. We consider two types of strong set-valued equilibrium problems as follows:

(SEP) Find $\bar{x} \in K$ such that for all $y \in K$,

$$F(\bar{x}, y) \subset C.$$

(WEP) Find $\bar{x} \in K$ such that for all $y \in K$,

$$F(\bar{x}, y) \cap C \neq \emptyset.$$

Now we assume that these problems suffer perturbation, which are expressed in terms of a perturbing parameter $\lambda \in \Lambda$ whenever Λ is a nonempty subset of a real topological vector space Z. This means that our problems are embedded into the following families, for $\lambda \in \Lambda$,

$(\mathrm{SEP})_\lambda$ Find $\bar{x} \in K(\lambda)$ such that for all $y \in K(\lambda)$,

$$F(\bar{x}, y, \lambda) \subset C,$$

$(\mathrm{WEP})_\lambda$ Find $\bar{x} \in K(\lambda)$ such that for all $y \in K(\lambda)$,

$$F(\bar{x}, y, \lambda) \cap C \neq \emptyset,$$

where $K : \Lambda \rightrightarrows X$ is a set-valued map with nonempty convex values, $F : A \times A \times \Lambda \subset X \times X \times Z \rightrightarrows Y$ is a set-valued map and $K(\Lambda) = \bigcup_{\lambda \in \Lambda} K(\lambda) \subset A$.

For $(\varepsilon, \lambda) \in \mathbb{R}_+ \times \Lambda$, where $\mathbb{R}_+$ is the set of nonnegative real numbers, the ε-approximate solution sets of $(\mathrm{SEP})_\lambda$ and $(\mathrm{WEP})_\lambda$ are denoted by

$$S(\varepsilon, \lambda) := \{x \in K(\lambda) \mid (F(x, y, \lambda) + \varepsilon e) \subset C \ \forall y \in K(\lambda)\}, \text{ and}$$
$$W(\varepsilon, \lambda) := \{x \in K(\lambda) \mid (F(x, y, \lambda) + \varepsilon e) \cap C \neq \emptyset \ \forall y \in K(\lambda)\},$$

respectively. We also consider auxiliary ε-approximate solution sets of the problems as below:

$$\widehat{S}(\varepsilon, \lambda) := \{x \in K(\lambda) \mid (F(x, y, \lambda) + \varepsilon e) \subset \mathrm{int}C \ \forall y \in K(\lambda)\}.$$
$$\widehat{W}(\varepsilon, \lambda) := \{x \in K(\lambda) \mid (F(x, y, \lambda) + \varepsilon e) \cap \mathrm{int}C \neq \emptyset \ \forall y \in K(\lambda)\}.$$

In this paper, we will investigate the stability in the sense of Hausdorff continuity of $S(\varepsilon, \lambda)$ and $W(\varepsilon, \lambda)$ as set-valued maps from $\mathbb{R}_+ \times \Lambda$ into X. Throughout this paper, we assume that the above solution sets are nonempty.

We now recall some basic notions needed in the sequel. Let $Q : X \rightrightarrows Y$ be a set-valued map.

Definition 2.1 (See [24, Definitions 1.4.1, 1.4.2, p. 38])

(a) Q is said to be upper semicontinuous (usc, shortly) at x_0 if for any neighborhood U of $Q(x_0)$, there is a neighborhood N of x_0 such that $Q(N) \subset U$.
(b) Q is said to be lower semicontinuous (lsc, shortly) at x_0 if for all $x_\alpha \to x_0$ and $y_0 \in Q(x_0)$, then there exist $y_\alpha \in Q(x_\alpha)$ such that $y_\alpha \to y_0$.
(c) Q is continuous at x_0 if it is both usc and lsc at x_0.

Definition 2.2 (See [25, Definition 2.5.12, p. 58])

(a) Q is said to be Hausdorff upper semicontinuous (H-usc, shortly) at x_0 if for each neighborhood $\mathbb{B}$ of the origin in Y, there exists a neighborhood N of x_0 such that $Q(x) \subset Q(x_0) + \mathbb{B}$ for all $x \in N$.
(b) Q is said to be Hausdorff lower semicontinuous (H-lsc, shortly) at x_0 if for each neighborhood $\mathbb{B}$ of the origin in Y, there exists a neighborhood N of x_0 such that $Q(x_0) \subset Q(x) + \mathbb{B}$ for all $x \in N$.
(c) Q is Hausdorff continuous at x_0 if it is both H-usc and H-lsc at x_0.

Definition 2.3 For $A \subset X$, a set-valued map $P : X \times X \times Z \rightrightarrows Y$ is said to be

(a) C-Hausdorff upper semicontinuous at λ_0 uniformly with respect to (wrt, shortly) $A \times A$ if for each neighborhood $\mathbb{B}$ of the origin in Y, there exists a neighborhood N of λ_0 such that for any $(x, y) \in A \times A$, we have

$$P(x, y, \lambda) \subset P(x, y, \lambda_0) + \mathbb{B} + C \quad \forall \lambda \in N.$$

(b) C-Hausdorff lower semicontinuous at λ_0 uniformly wrt $A \times A$ if for each neighborhood $\mathbb{B}$ of the origin in Y, there exists a neighborhood N of λ_0 such that for any $(x, y) \in A \times A$, we have

$$P(x, y, \lambda_0) \subset P(x, y, \lambda) + \mathbb{B} + C \quad \forall \lambda \in N.$$

(c) C-Hausdorff continuous at λ_0 uniformly wrt $A \times A$ if it is both C-Hausdorff upper and lower semicontinuous at λ_0 uniformly wrt $A \times A$.

Definition 2.4 (See [26, Definition 2.1]) Q is said to be C-convex on a convex subset $A \subset X$ if for any $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$tQ(x_1) + (1-t)Q(x_2) \subset Q(tx_1 + (1-t)x_2) + C, \tag{1}$$

and Q is said to be C-concave if (1) is replaced by

$$Q(tx_1 + (1-t)x_2) \subset tQ(x_1) + (1-t)Q(x_2) + C.$$

Next, we propose a generalized concept related to the C-concavity.

Definition 2.5 Let $\Omega \subset \mathbb{R}$, $F : X \rightrightarrows Y$ be a set-valued map and A be a convex subset of X.

(a) F is said to be Ω-concave wrt e of the first type on A if for any $x_1, x_2 \in A$, $r_1, r_2 \in \Omega$ such that $F(x_1) \subset r_1 e + C$ and $F(x_2) \subset r_2 e + \mathrm{int}C$, then $F(tx_1 + (1-t)x_2) \subset [tr_1 + (1-t)r_2]e + \mathrm{int}C$ for all $t \in]0, 1[$.
(b) F is said to be Ω-concave wrt e of the second type on A if for any $x_1, x_2 \in A$, $r_1, r_2 \in \Omega$ such that $F(x_1) \cap (r_1 e + C) \neq \emptyset$ and $F(x_2) \cap (r_2 e + \mathrm{int}C) \neq \emptyset$, then $F(tx_1 + (1-t)x_2) \cap ([tr_1 + (1-t)r_2]e + \mathrm{int}C) \neq \emptyset$ for all $t \in]0, 1[$.

Remark 2.1 (a) It follows from Definition 2.5 that if a map satisfies the Ω-concavity properties on A, then it is $\widehat{\Omega}$-concave on A for all $\widehat{\Omega} \subset \Omega$.
(b) In the case that F is a single-valued map, then the Ω-concavity wrt e of the first and second types on A are coincident, say the Ω-concavity wrt e on A, and they are a relaxation of the C-concavity. Obviously, the Dirichlet function, $D(x) = 1$ for x rational and $D(x) = 0$ for x irrational, is $-\mathbb{R}_+$-concave on $\mathbb{R}$ wrt $e = 1$ and $C = \mathbb{R}_+$ but it is not concave on $\mathbb{R}$, and so the Ω-concavity concept of a real function is a weakened version of the classical one.

Lemma 2.1 *Let F, A be as in* Definition 2.5. *The following statements hold.*

(a) *If F is C-concave on A, then F is $\mathbb{R}$-concave wrt e of the first type on A.*
(b) *If F is $-C$-convex on A, then F is $\mathbb{R}$-concave wrt e of the second type on A.*

Proof (a) Let $x_1, x_2 \in A$, $r_1, r_2 \in \mathbb{R}$, $F(x_1) \subset r_1 e + C$ and $F(x_2) \subset r_2 e + \mathrm{int}C$. Then, for each $t \in]0, 1[$, the C-concavity of F gives us

$$\begin{aligned} F(tx_1 + (1-t)x_2) &\subset tF(x_1) + (1-t)F(x_2) + C \\ &\subset (tr_1 + (1-t)r_2))e + tC + (1-t)\mathrm{int}C + C \\ &\subset (tr_1 + (1-t)r_2))e + \mathrm{int}C. \end{aligned}$$

Hence, F is $\mathbb{R}$-concave wrt e of the first type on A.

(b) Let $x_1, x_2 \in A, r_1, r_2 \in \mathbb{R}$, $F(x_1) \cap (r_1 e + C) \neq \emptyset$ and $F(x_2) \cap (r_2 e + \mathrm{int}C) \neq \emptyset$. For each $z_1 \in F(x_1) \cap (r_1 e + C)$, $z_2 \in F(x_2) \cap (r_2 e + \mathrm{int}C)$ and $t \in]0, 1[$, the $-C$-convexity of F leads to the existence of $z_t \in F(tx_1 + (1-t)x_2)$ and $c_t \in C$ satisfying $z_t = tz_1 + (1-t)z_2 + c_t$. Besides, there are $c_1 \in C$ and $c_2 \in \mathrm{int}C$ such that $z_1 = r_1 e + c_1$ and $z_2 = r_2 e + c_2$. Therefore,

$$z_t = t(r_1 e + c_1) + (1-t)(r_2 e + c_2) + c_t \in tr_1 e + (1-t)r_2 e + \mathrm{int}C.$$

So, $F(tx_1 + (1-t)x_2) \cap ((tr_1 + (1-t)r_2))e + \mathrm{int}C) \neq \emptyset$. □

The following example shows that the converses of Lemma 2.1 are not true.

Example 2.1 Let $X = A = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $e = (1, 1)$ and

$$F(x) = \begin{cases} \{(3, 1)\}, & \text{if } x \neq 0, \\ \{(1, 1)\}, & \text{if } x = 0. \end{cases}$$

Then, for $x_1, x_2 \in \mathbb{R}$ and $r_1, r_2 \in \mathbb{R}$ such that $F(x_1) \in r_1 e + C$ and $F(x_2) \in r_2 e + \text{int}C$. One has $r_1 \leq 1$ and $r_2 < 1$. So, for all $t \in]0, 1[$, $F(tx_1 + (1-t)x_2) \in (tr_1 + (1-t)r_2)e + C$. Hence, F is $\mathbb{R}$-concave wrt e on $\mathbb{R}$. However, let $x_1 = -1, x_2 = 1$ and $t = \frac{1}{2}$, $F\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = F(0) = (1, 1) \notin \frac{1}{2}F(-1) + \frac{1}{2}F(1) + C = (3, 1) + C$, and hence F is not concave on $\mathbb{R}$.

3 Continuity of approximate solution maps to strong set-valued equilibrium problems

In this section we mainly discuss the Hausdorff continuity of the solution maps S and W to $(\text{SEP})_\lambda$ and $(\text{WEP})_\lambda$, respectively. First, we state a result on the Hausdorff continuity of the map S.

Theorem 3.1 *For* $(\text{SEP})_\lambda$, *let* $S(\varepsilon, \lambda)$ *be nonempty in a neighborhood of the reference point* $(\varepsilon_0, \lambda_0) \in]0, +\infty[\times \Lambda$. *Assume further that the following conditions hold*:

(i) *K is continuous and compact-valued at λ_0 and $K(\Lambda)$ is bounded*;

(ii) *there exists a neighborhood N of λ_0 such that F is continuous and compact-valued on $K(N) \times K(N) \times \{\lambda_0\}$*;

(iii) *for all $\lambda \in \Lambda$ and $y \in K(\lambda)$, $F(\cdot, y, \lambda)$ is $-\mathbb{R}_+$-concave wrt e of the first type on $K(\lambda)$*.

Then, the map S is Hausdorff continuous at $(\varepsilon_0, \lambda_0)$.

Proof For the sake of convenience we divide the proof into three steps.

Step 1. We show that for each closed convex neighborhood $\mathbb{B}$ of the origin in X, there exists a neighborhood V of ε_0 such that for all $\lambda \in \Lambda$ and $\varepsilon \in V$,

$$S(\varepsilon, \lambda) \subset S(\varepsilon_0, \lambda) + \mathbb{B} \text{ and } S(\varepsilon_0, \lambda) \subset S(\varepsilon, \lambda) + \mathbb{B}. \tag{2}$$

It follows from the boundedness of $K(\Lambda)$ that for each closed convex neighborhood $\mathbb{B}$ of origin in X, there exists $\rho > 0$ satisfying

$$K(\Lambda) - K(\Lambda) \subset \rho\mathbb{B}. \tag{3}$$

For every $\eta \in (0, \varepsilon_0)$ and $\varepsilon_1, \varepsilon_2 \in [\varepsilon_0 - \eta, \varepsilon_0 + \eta]$ with $\varepsilon_1 < \varepsilon_2$ and $x_1 \in S(\varepsilon_1, \lambda)$, we have $F(x_1, y, \lambda) + \varepsilon_1 e \subset C$ for all $y \in K(\lambda)$. We can write $F(x_1, y, \lambda) + \varepsilon_2 e = (F(x_1, y, \lambda) + \varepsilon_1 e) + (\varepsilon_2 - \varepsilon_1)e \subset C$, i.e., $x_1 \in S(\varepsilon_2, \lambda)$. Hence, $S(\varepsilon_1, \lambda) \subset S(\varepsilon_2, \lambda)$. This implies that for all $\varepsilon \in [\varepsilon_0 - \eta, \varepsilon_0 + \eta]$,

$$S(\varepsilon_0 - \eta, \lambda) \subset S(\varepsilon, \lambda) \subset S(\varepsilon_0 + \eta, \lambda). \tag{4}$$

For any θ with $1 < \theta < \dfrac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1}$ and $\gamma := \varepsilon_2 + \theta(\varepsilon_1 - \varepsilon_2)$, it is claimed that

$$\frac{1}{\theta}\widehat{S}(\gamma, \lambda) + \left(1 - \frac{1}{\theta}\right) S(\varepsilon_2, \lambda) \subset S(\varepsilon_1, \lambda). \tag{5}$$

In fact, thanks to the convexity of $K(\lambda)$, we have $x_\theta := \frac{1}{\theta}x_1 + \left(1 - \frac{1}{\theta}\right)x_2 \in K(\lambda)$ for all $x_1 \in \widehat{S}(\gamma, \lambda)$ and $x_2 \in S(\varepsilon_2, \lambda)$. This together with the $-\mathbb{R}_+$-concavity of F implies that, for each $y \in K(\lambda)$,

$$F(x_\theta, y, \lambda) + \varepsilon_1 e = F(x_\theta, y, \lambda) + \frac{1}{\theta}\gamma e + \left(1 - \frac{1}{\theta}\right)\varepsilon_2 e \subset \operatorname{int}C \subset C,$$

that is, (5) is proved. It follows from (5) that

$$\left(1 - \frac{1}{\theta}\right) S(\varepsilon_2, \lambda) \subset S(\varepsilon_1, \lambda) - \frac{1}{\theta}\widehat{S}(\gamma, \lambda),$$

which yields that

$$\begin{aligned} S(\varepsilon_2, \lambda) &\subset S(\varepsilon_1, \lambda) + \frac{1}{\theta - 1}\left[S(\varepsilon_1, \lambda) - \widehat{S}(\gamma, \lambda)\right] \\ &\subset S(\varepsilon_1, \lambda) + \frac{1}{\theta - 1}[K(\lambda) - K(\lambda)]. \end{aligned}$$

Combining this with (3), we get

$$S(\varepsilon_2, \lambda) \subset S(\varepsilon_1, \lambda) + \frac{\rho}{\theta - 1}\mathbb{B}. \tag{6}$$

Choosing η_0 such that $\eta_0 < \frac{\varepsilon_0}{\rho+1}$ and considering $V = [\varepsilon_0 - \eta_0, \varepsilon_0 + \eta_0]$ as a neighborhood of ε_0. Substituting $\varepsilon_2 = \varepsilon_0$, $\varepsilon_1 = \varepsilon_0 - \eta_0$ and $\theta = \rho + 1$ (clearly, $1 < \theta < \frac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1}$) in (6), we get $S(\varepsilon_0, \lambda) \subset S(\varepsilon_0 - \eta_0, \lambda) + \mathbb{B}$. This together with (4) implies that for all $\varepsilon \in V$, $S(\varepsilon_0, \lambda) \subset S(\varepsilon, \lambda) + \mathbb{B}$.

Analogously, substituting $\varepsilon_2 = \varepsilon_0 + \eta_0$, $\varepsilon_1 = \varepsilon_0$ and $\theta = \rho + 1$ in (6) and combining this with (4), we have $S(\varepsilon, \lambda) \subset S(\varepsilon_0 + \eta_0, \lambda) \subset S(\varepsilon_0, \lambda) + \mathbb{B}$. Therefore, (2) is examined.

Step 2. For any $\varepsilon \in V$, we prove that $S(\varepsilon, \cdot)$ is Hausdorff continuous at λ_0.

By [27, Proposition 3.1(i)], we show that $S(\varepsilon, \cdot)$ is H-usc at λ_0 by proving $S(\varepsilon, \cdot)$ is usc at λ_0. Suppose that $S(\varepsilon, \cdot)$ is not usc at λ_0. Then, we can find a neighborhood U of $S(\varepsilon, \lambda_0)$, a net $\{\lambda_\alpha\}$ converging to λ_0, and $x_\alpha \in S(\varepsilon, \lambda_\alpha)$ but $x_\alpha \notin U$ for all α. By [28, Proposition 2.19, p. 41], the upper semicontinuity of K at λ_0 and the compactness of $K(\lambda_0)$ allow us to assume that there is $x_0 \in K(\lambda_0)$ with $x_\alpha \to x_0$. If $x_0 \notin S(\varepsilon, \lambda_0)$, then there are $y_0 \in K(\lambda_0)$ and $z_0 \in F(x_0, y_0, \lambda_0)$ such that $z_0 + \varepsilon e \notin C$. By [28, Proposition 2.6, p. 37], the lower semicontinuity of K at λ_0 shows the existence of $y_\alpha \in K(\lambda_\alpha)$ such that $y_\alpha \to y_0$. Since $x_\alpha \in S(\varepsilon, \lambda_\alpha)$, for all $z_\alpha \in F(x_\alpha, y_\alpha, \lambda_\alpha)$, we get $z_\alpha + \varepsilon e \in C$, which together with the lower semicontinuity of F implies that there is $\bar{z}_\alpha \in F(x_\alpha, y_\alpha, \lambda_\alpha)$ with $\bar{z}_\alpha \to z_0$ such that $\bar{z}_\alpha + \varepsilon e \in C$. That is, $z_0 + \varepsilon e \in C$ as C is closed, which arrives at again contradiction as $z_0 + \varepsilon e \notin C$.

We now prove the Hausdorff lower semicontinuity of $S(\varepsilon, \cdot)$ at λ_0. By [28, Theorem 2.68, p. 62], it suffices to check $S(\varepsilon, \cdot)$ is lsc at λ_0 and $S(\varepsilon, \lambda_0)$ is compact. Let us start by showing the lower semicontinuity of $\widehat{S}(\varepsilon, \cdot)$. Suppose that $\widehat{S}(\varepsilon, \cdot)$ is not lsc at λ_0, i.e., there are $x_0 \in \widehat{S}(\varepsilon, \lambda_0)$ and a net $\{\lambda_\alpha\}$ converging to λ_0, for all $x_\alpha \in \widehat{S}(\varepsilon, \lambda_\alpha)$, $x_\alpha \nrightarrow x_0$. Thanks to the lower semicontinuity of K at λ_0, there exists $\bar{x}_\alpha \in K(\lambda_\alpha)$ satisfying $\bar{x}_\alpha \to x_0$. Due to above contradiction assumption, there must be a subnet $\{\bar{x}_\beta\}$ of $\bar{x}_\alpha$ such that for all β, $\bar{x}_\beta \notin \widehat{S}(\varepsilon, \lambda_\beta)$, i.e., there exist $y_\beta \in K(\lambda_\beta)$ and $z_\beta \in F(\bar{x}_\beta, y_\beta, \lambda_\beta)$, $z_\beta + \varepsilon e \notin \operatorname{int} C$. Since K is usc and compact-valued at λ_0, one has $y_0 \in K(\lambda_0)$ with $y_\beta \to y_0$ (take a subnet if necessary). By (ii), there is $z_0 \in F(x_0, y_0, \lambda_0)$ with $z_\beta \to z_0$ such that $z_0 + \varepsilon e \notin \operatorname{int} C$, which is impossible as x_0 belongs to $\widehat{S}(\varepsilon, \lambda_0)$. Hence, $\widehat{S}(\varepsilon, \cdot)$ is lsc at λ_0. Now, we claim that

$$S(\varepsilon, \lambda_0) \subset \operatorname{cl}\widehat{S}(\varepsilon, \lambda_0), \tag{7}$$

where "cl" stands for the closure. Let $\bar{x} \in S(\varepsilon, \lambda_0)$, $x_1 \in \widehat{S}(\varepsilon, \lambda_0)$, $t \in]0, 1[$ and $y \in K(\lambda_0)$, by the $-\mathbb{R}_+$-concavity of $F(\cdot, y, \lambda_0)$, one has $F(tx_1 + (1-t)\bar{x}, y, \lambda_0) + \varepsilon e \subset \operatorname{int} C$, and so $x_t := tx_1 + (1-t)\bar{x} \in \widehat{S}(\varepsilon, \lambda_0)$. Because $x_t \to \bar{x}$ when $t \to 0$, $\bar{x}$ belongs to $\operatorname{cl}\widehat{S}(\varepsilon, \lambda_0)$, and hence (7) follows. Also, by [28, Proposition 2.6, p. 37], the lower semicontinuity at λ_0 of $\widehat{S}(\varepsilon, \cdot)$ leads to

$$S(\varepsilon, \lambda_0) \subset \operatorname{cl}\widehat{S}(\varepsilon, \lambda_0) \subset \liminf \widehat{S}(\varepsilon, \lambda_\alpha) \subset \liminf S(\varepsilon, \lambda_\alpha),$$

i.e., $S(\varepsilon, \cdot)$ is lsc at λ_0.

For the compactness of $S(\varepsilon, \lambda_0)$, it is enough to show that $S(\varepsilon, \lambda_0)$ is closed in $K(\lambda_0)$. Let $x_\alpha \in S(\varepsilon, \lambda_0)$ with $x_\alpha \to x_0$, then $x_\alpha \in K(\lambda_0)$ and $x_0 \in K(\lambda_0)$. For each $y \in K(\lambda_0)$, we have $z_\alpha + \varepsilon e \in C$, for all $z_\alpha \in F(x_\alpha, y, \lambda_0)$. Thanks to (ii) and the closedness of C, we get $z_0 + \varepsilon e \in C$, for all $z_0 \in F(x_0, y, \lambda_0)$, i.e., $x_0 \in S(\varepsilon, \lambda_0)$.

Step 3. We are ready to complete the proof. For each neighborhood $\mathbb{B}$ of the origin in X, we can find a neighborhood $\mathbb{B}_1$ of the origin in X such that

$$\mathbb{B}_1 + \mathbb{B}_1 \subset \mathbb{B}. \tag{8}$$

By Step 1, for any neighborhood $\mathbb{B}_1$, there is a neighnorhood V_1 of ε_0 such that for all $\lambda \in \Lambda$,

$$S(\varepsilon, \lambda) \subset S(\varepsilon_0, \lambda) + \mathbb{B}_1 \text{ and } S(\varepsilon_0, \lambda) \subset S(\varepsilon, \lambda) + \mathbb{B}_1. \tag{9}$$

For each $\varepsilon \in V_1$, since $S(\varepsilon, \cdot)$ is H-continuous at λ_0, there exists a neighborhood N_1 of λ_0 satisfying, for all $\lambda \in N_1$,

$$S(\varepsilon_0, \lambda) \subset S(\varepsilon_0, \lambda_0) + \mathbb{B}_1 \text{ and } S(\varepsilon_0, \lambda_0) \subset S(\varepsilon_0, \lambda) + \mathbb{B}_1. \tag{10}$$

It is clear that $V_1 \times N_1$ is a neighborhood of $(\varepsilon_0, \lambda_0)$. Combining this with (8), (9) and (10), we have, for all $(\varepsilon, \lambda) \in V_1 \times N_1$,

$$S(\varepsilon, \lambda) \subset S(\varepsilon_0, \lambda_0) + \mathbb{B} \text{ and } S(\varepsilon_0, \lambda_0) \subset S(\varepsilon, \lambda) + \mathbb{B}.$$

Therefore, S is H-continuous at $(\varepsilon_0, \lambda_0)$. □

The following example illustrate the applicability of Theorem 3.1.

Example 3.1 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $e = 1 \in \text{int}C$, $\Lambda = [0, 1]$, $K(\lambda) = [-1, \lambda]$ and

$$F(x, y, \lambda) = \begin{cases} \left[-x^3 + y, y + \pi - \cos\lambda\right], & \text{if } x < 0, \\ [-x + y, y + \pi - \cos\lambda], & \text{if } x \geq 0. \end{cases}$$

Clearly, the assumptions of Theorem 3.1 are satisfied. Applying this theorem, we get the Hausdorff continuity of the solution map to this problem. (Indeed, direct computations give us

$$S(\varepsilon, \lambda) = \begin{cases} [-1, \sqrt[3]{\varepsilon - 1}], & \text{if} \varepsilon < 1, \\ [-1, \min\{\lambda, \varepsilon - 1\}], & \text{if} \varepsilon \geq 1, \end{cases}$$

which is Hausdorff continuous).

We now derive an example to show that the assumption (iii) in Theorem 3.1 is essential.

Example 3.2 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $e = 1 \in \text{int}C$, $\Lambda = [0, 2]$, $K(\lambda) = [\lambda - 1, \lambda + 4]$, $\varepsilon_0 = 1$, $\lambda_0 = 1$ and $F(x, y, \lambda) = \{x^2 - 2x + y - \lambda\}$. It is easy to verify that conditions of Theorem 3.1 are satisfied, except for the $-\mathbb{R}_+$-concavity. For $(\varepsilon, \lambda) \in \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$, we get

$$\begin{aligned} S(\varepsilon, \lambda) &= \{x \in [\lambda - 1, \lambda + 4] \mid x^2 - 2x + y - \lambda + \varepsilon \geq 0 \ \forall y \in [\lambda - 1, \lambda + 4]\} \\ &= \{x \in [\lambda - 1, \lambda + 4] \mid x^2 - 2x - 1 + \varepsilon \geq 0\} \\ &= \left([\lambda - 1, \lambda + 4] \cap] - \infty, 1 - \sqrt{2 - \varepsilon}]\right) \cup [1 + \sqrt{2 - \varepsilon}, \lambda + 4] \neq \emptyset. \end{aligned}$$

Let $x_0 = 0 \in S(\varepsilon_0, \lambda_0) = \{0\} \cup [2, 5]$ and $(\varepsilon_n, \lambda_n) := \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) \to (\varepsilon_0, \lambda_0)$, then for all $x_n \in S(\varepsilon_n, \lambda_n) = \left[1 + \sqrt{1 + \frac{1}{n}}, 5 + \frac{1}{n}\right]$, $\{x_n\}$ cannot converge to x_0. Therefore, S is not even lower semicontinuous at $(\varepsilon_0, \lambda_0)$.

Using the same techniques of the proof for Theorem 3.1, we also obtain the following result.

Theorem 3.2 *For* (WEP)$_\lambda$, *assume that the existence of solutions and* (i), (ii) *are as in* Theorem 3.1 *and replace* (iii) *by*

(iii′) for all $\lambda \in \Lambda$ and $y \in K(\lambda)$, $F(\cdot, y, \lambda)$ is $-\mathbb{R}_+$-concave wrt e of the second type on $K(\lambda)$.

Then, the map W is Hausdorff continuous at $(\varepsilon_0, \lambda_0)$.

In recent years, there have been a lot of works in the literature dealing with mathematical models with optimization/equilibrium constraints [29–32], which leads to consider a class of equilibrium problems with nonparametric constraints (see [18,20,21]). In the rest of this section, we consider the case of $K(\lambda) \equiv K$ for all λ, where K is a nonempty compact and convex subset. For this special case, we obtain very beautiful results, which cannot be derived from the previous results.

Theorem 3.3 *Assume that the solution sets to* (SEP)$_\lambda$ *are nonempty in a neighborhood of the reference point* $(\varepsilon_0, \lambda_0) \in]0, +\infty[\times \Lambda$. *Assume further that*

(i) *F is C-Hausdorff continuous at λ_0 uniformly wrt $K \times K$;*
(ii) *for each $y \in K$ and $\lambda \in \Lambda$, $F(\cdot, y, \lambda)$ is $-\mathbb{R}_+$-concave wrt e of the first type on K.*

Then, the map S is Hausdorff continuous at $(\varepsilon_0, \lambda_0)$.

Proof We can retain Step 1 of the proof of Theorem 3.1, which only employs the $-\mathbb{R}_+$-concavity of F, and hence the inclusion (2) holds.

Setting $U := \left\{u \in Y \mid u \in -\frac{\eta_0}{2}e + C\right\}$ where η_0 is defined as in the proof of Theorem 3.1, then U is a neighborhood of the origin of Y and $U + (\varepsilon - \varepsilon_0 + \eta_0)e \subset C$ with $\varepsilon \in \left[\varepsilon_0 - \frac{\eta_0}{2}, \varepsilon_0 + \frac{\eta_0}{2}\right]$. The C-Hausdorff upper semicontinuity uniformly wrt $K \times K$ of F leads to the existence of a neighborhood N_1 of λ_0 such that $F(x, y, \lambda) \subset F(x, y, \lambda_0) + U + C$ for all $x, y \in K$ and $\lambda \in N_1$. For each $x_0 \in S(\varepsilon_0 - \eta_0, \lambda_0)$, by the convexity of C, one has

$$F(x_0, y, \lambda) + \varepsilon e \subset F(x_0, y, \lambda_0) + (\varepsilon_0 - \eta_0)e + U + (\varepsilon - \varepsilon_0 + \eta_0)e + C \subset C.$$

So, $S(\varepsilon_0 - \eta_0, \lambda_0) \subset S(\varepsilon, \lambda)$. This together with (2) implies that

$$S(\varepsilon_0, \lambda_0) \subset S(\varepsilon_0 - \eta_0, \lambda_0) + \mathbb{B} \subset S(\varepsilon, \lambda) + \mathbb{B}. \tag{11}$$

Similarly, we get $S(\varepsilon, \lambda) \subset S(\varepsilon_0 + \eta_0, \lambda_0) \subset S(\varepsilon_0, \lambda_0) + \mathbb{B}$. Combine this and (11), we get the conlusion of Theorem 3.3. □

Remark 3.1 Because the continuity property of F in the first and the second components are not assumed, so techniques of the proof for Theorem 3.3 are strictly different from those for Theorem 3.1, and hence Theorem 3.3 is not a corollary of Theorem 3.1. Of course, we hope that such techniques can apply to the proof of Theorem 3.1 in order to weaken the mentioned assumptions for the case of parametric constraints. The following example gives a suggestion that it seems to be unable to realize our desire.

Example 3.3 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $e = 1 \in \operatorname{int}C$, $\Lambda = [0, 1]$, $K(\lambda) = [\lambda, 2 - \lambda]$, $\varepsilon_0 = \frac{1}{2}$, $\lambda_0 = 0$ and

$$F(x, y, \lambda) = \begin{cases} \{y^2 + \lambda^2\}, & \text{if } x = 0, \\ \{-x + \lambda\}, & \text{if } 0 < x < 2, \\ \{-\lambda\}, & \text{if } x = 2. \end{cases}$$

Obviously, the conditions of Theorem 3.1 are satisfied, except for the continuity of F in the first component. Clearly, $x_0 = 2 \in S(\varepsilon_0, \lambda_0) = [0, \frac{1}{2}] \cup \{2\}$ and $(\varepsilon_n, \lambda_n) := \left(\frac{1}{2} - \frac{1}{n}, \frac{1}{n}\right) \to (\varepsilon_0, \lambda_0)$. However, for all $x_n \in S(\varepsilon_n, \lambda_n) = \left[\frac{1}{n}, \frac{1}{2}\right]$, $\{x_n\}$ does not converge to $x_0 = 2$. So, S is not even lsc at $(\varepsilon_0, \lambda_0)$.

For $(\mathrm{WEP})_\lambda$, we also have a similar result to that of Theorem 3.3.

Theorem 3.4 *Assume that* $(\mathrm{WEP})_\lambda$ *is solvable on a neighborhood of the reference point* $(\varepsilon_0, \lambda_0) \in]0, +\infty[\times \Lambda$. *Assume further that*

(i) *F is C-Hausdorff continuous at λ_0 uniformly wrt $K \times K$;*
(ii) *for each $y \in K$ and $\lambda \in \Lambda$, $F(\cdot, y, \lambda)$ is $-\mathbb{R}_+$-concave wrt e of the second type on K.*

Then, the map W is Hausdorff continuous at $(\varepsilon_0, \lambda_0)$.

When $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $e = 1$, and $F : X \to \mathbb{R}$ is a real function, $(\mathrm{SEP})_\lambda$ and $(\mathrm{WEP})_\lambda$ reduce to a scalar equilibrium problem (EP) considered in [18]. The following result is directly derived from Theorems 3.3 and 3.4.

Corollary 3.1 *Assume that* (EP) *is solvable on a neighborhood of the reference point* $(\varepsilon_0, \lambda_0) \in]0, +\infty[\times \Lambda$. *Assume further that*

(i) *F is C-Hausdorff continuous at λ_0 uniformly wrt $K \times K$;*
(ii) *for each $y \in K$ and $\lambda \in \Lambda$, $F(\cdot, y, \lambda)$ is $-\mathbb{R}_+$-concave on K.*

Then, the solution map S_0 to (EP) *is Hausdorff continuous at* $(\varepsilon_0, \lambda_0)$.

Remark 3.2 In this special case, Corollary 3.1 is an improvement of Theorem 3.1 in [18], namely, the continuity of F in the first component is not utilized, and the concavity of F is relaxed to the $-\mathbb{R}_+$-concavity.

The following example is given to illustrate a case in which Corollary 3.1 can apply while Theorem 3.1 in [18] cannot.

Example 3.4 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $e = 1$, $\Lambda = [0, 1]$, $K = [0, 3]$ and

$$F(x, y, \lambda) = \begin{cases} \{\pi^{\lambda+y} x^2\}, & \text{if } x \leq 2, \\ \{\pi^{\lambda+y}(-15x + 42)\}, & \text{if } x > 2. \end{cases}$$

It is not hard to check that all assumptions of Corollary 3.1 are satisfied. (By direct computations, we have $S_0(\varepsilon, \lambda) = \left[0, \frac{42\pi^\lambda + \varepsilon}{15\pi^\lambda}\right]$ for all $(\varepsilon, \lambda) \in]0, +\infty[\times \Lambda$, so S_0 is Hausdorff continuous). However, Theorem 3.1 in [18] does not work because F is neither continuous nor concave in the first component on K.

4 Browder variational inclusions

In this section we present applications of our main results in Sect. 3. Namely, the Hausdorff continuity of solution maps to two versions of Browder variational inclusions is derived. To the best of our knowledge, there is no work in the literature devoted to the mentioned stability for these problems.

It is known that the Browder variational inclusions have the form of finding $x_0 \in K(\lambda)$ such that $\langle \Omega, x_0 \rangle \subset \mathbb{R}_+$ or finding $x_0 \in K(\lambda)$ such that $\langle \Omega, x_0 \rangle \cap \mathbb{R}_+ \neq \emptyset$, where X^* is the dual space of X, $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X, Ω is a subset of X^*, and

$$\langle \Omega, x \rangle := \{\langle x^*, x \rangle \mid x^* \in \Omega\}.$$

Definition 4.1 (See [28, Definition 1.1, p. 302]) A set-valued map $G : X \rightrightarrows X^*$ is said to be monotone on $A \subset X$ if for all $(x_1, z_1), (x_2, z_2) \in \text{Graph}G := \{(x, z) \mid z \in G(x)\}$, $\langle z_1 - z_2, x_1 - x_2 \rangle \geq 0$.

Definition 4.2 (See [24, Definition 2.1.1, p. 56]) A set-valued map $G : X \rightrightarrows Y$ is convex on a convex subset A of X if Graph$G$ is a convex set $A \times Y$.

Remark 4.1 (a) G is convex on A if and only if for all $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$tG(x_1) + (1 - t)G(x_2) \subset G(tx_1 + (1 - t)x_2).$$

It follows from this, G is said to be concave on A if for all $x_1, x_2 \in A$ and $t \in [0, 1]$, $G(tx_1 + (1 - t)x_2) \subset tG(x_1) + (1 - t)G(x_2)$.

(b) G is convex if and only if it is C-convex for any pointed cones C.

Corollary 4.1 *Assume that*

(i) *K is continuous, compact-valued at λ_0 and $K(\Lambda)$ is bounded*;
(ii) *there exists a neighborhood N of λ_0 such that $G : K(\lambda) \times \Lambda \rightrightarrows X^*$ is continuous and compact-valued on $K(N) \times \{\lambda_0\}$*;
(iii) *for all $\lambda \in \Lambda$ and $y \in K(\lambda)$, $G(\cdot, \lambda)$ is monotone and concave on $K(\lambda)$.*

Then, the approximate solution map

$$(\varepsilon, \lambda) \mapsto S^s(\varepsilon, \lambda) := \{x \in K(\lambda) \mid \langle G(x, \lambda), y - x \rangle + \varepsilon \subset \mathbb{R}_+ \, \forall y \in K(\lambda)\}$$

is Hausdorff continuous at a reference point $(\varepsilon_0, \lambda_0)$.

Proof Setting $F(x, y, \lambda) := \langle G(x, \lambda), y - x \rangle$ and $C = \mathbb{R}_+$, we will prove this corollary by checking all assumptions of Theorem 3.1.

Obviously, (i) and (ii) of Theorem 3.1 is fulfilled. Assumption (iii) of Theorem 3.1 is satisfied. In fact, let $x_1, x_2 \in K(\Lambda)$, $t \in [0, 1]$, $r_1, r_2 \in -\mathbb{R}_+$ and $y \in K(\Lambda)$, we need to show that if $F(x_1, y, \lambda_0) \subset r_1 + C$ and $F(x_2, y, \lambda_0) \subset r_2 + \text{int}C$, then

$$F(tx_1 + (1 - t)x_2, y, \lambda_0) \subset tr_1 + (1 - t)r_2 + \text{int}C. \quad (12)$$

For each $z \in F(tx_1 + (1 - t)x_2, y, \lambda_0) = \langle G(tx_1 + (1 - t)x_2, \lambda_0), y - tx_1 - (1 - t)x_2 \rangle$, the concavity of $G(\cdot, \lambda_0)$ leads to the existence of $z_1 \in G(x_1, \lambda_0)$ and $z_2 \in G(x_2, \lambda_0)$

such that $z = \langle tz_1 + (1-t)z_2, y - tx_1 - (1-t)x_2\rangle$. It follows that

$$\begin{aligned} z &= t^2\langle z_1, y - x_1\rangle + (1-t)^2\langle z_2, y - x_2\rangle + t(1-t)(\langle z_1, y - x_2\rangle + \langle z_2, y - x_1\rangle) \\ &\quad - t\langle z_1, y - x_1\rangle - (1-t)\langle z_2, y - x_2\rangle + t\langle z_1, y - x_1\rangle + (1-t)\langle z_2, y - x_2\rangle \\ &= t(1-t)(\langle -z_1, y - x_1\rangle + \langle -z_2, y - x_2\rangle + \langle z_1, y - x_2\rangle + \langle z_2, y - x_1\rangle) \\ &\quad + t\langle z_1, y - x_1\rangle + (1-t)\langle z_2, y - x_2\rangle \\ &= t\langle z_1, y - x_1\rangle + (1-t)\langle z_2, y - x_2\rangle + t(1-t)\langle z_2 - z_1, x_2 - x_1\rangle \\ &\in tF(x_1, y, \lambda_0) + (1-t)F(x_2, y, \lambda_0) + C \subset tr_1 + (1-t)r_2 + \text{int}C, \end{aligned}$$

which yields (12). The proof is complete. □

Corollary 4.2 *Assume that* (i) *and* (ii) *are as in* Corollary 4.1 *and replace* (iii) *by the following condition:*

(iii′) for all $\lambda \in \Lambda$ and $y \in K(\lambda)$, $G(\cdot, \lambda)$ is monotone and convex on $K(\lambda)$.

Then we have the same conclusion as of Corollary 4.1 *for the following approximate solution map*

$$(\varepsilon, \lambda) \mapsto S^w(\varepsilon, \lambda) := \{x \in K(\lambda) \mid (\langle G(x, \lambda), y - x\rangle + \varepsilon) \cap \mathbb{R}_+ \neq \emptyset \ \forall y \in K(\lambda)\}.$$

Proof It can be proved similarly to that of Corollary 4.1. □

Acknowledgements The authors would like to thank two anonymous referees for their valuable remarks and suggestions that helped us significantly the presentation of paper. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.01-2020.11.

References

1. Alleche, B.: Multivalued mixed variational inequalities with locally Lipschitzian and locally cocoercive multivalued mappings. J. Math. Anal. Appl. **399**(2), 625–637 (2013)
2. Shih, M.H., Tan, K.K.: Browder–Hartman–Stampacchia variational inequalities for multi-valued monotone operators. J. Math. Anal. Appl. **134**(2), 431–440 (1988)
3. Kassay, G., Rădulescu, V.: Equilibrium Problems and Applications. Academic Press, Cambridge (2018)
4. Abbasi, M., Rezaei, M.: Existence results for approximate set-valued equilibrium problems. Adv. Oper. Theory **1**(2), 189–205 (2016)
5. Alleche, B., Rădulescu, V.D.: Set-valued equilibrium problems with applications to Browder variational inclusions and to fixed point theory. Nonlinear Anal. Real World Appl. **28**, 251–268 (2016)
6. Alleche, B., Rădulescu, V.D.: Further on set-valued equilibrium problems and applications to Browder variational inclusions. J. Optim. Theory Appl. **175**(1), 39–58 (2017)
7. Kristaly, A., Varga, C.: Set-valued versions of Ky Fan's inequality with application to variational inclusion theory. J. Math. Anal. Appl. **282**(1), 8–20 (2003)
8. Song, W.: Vector equilibrium problems with set-valued mappings. In: Giannessi, F. (ed.) Vector Variational Inequalities and Vector Equilibria: Nonconvex Optimization and Its Applications, vol. 38, pp. 403–422. Springer, Boston (2000)
9. Li, S.J., Liu, H.M., Chen, C.R.: Lower semicontinuity of parametric generalized weak vector equilibrium problems. Bull. Aust. Math. Soc. **81**(1), 85–95 (2010)
10. Chen, B., Gong, X.H.: Continuity of the solution set to parametric set-valued weak vector equilibrium problems. Pac. J. Optim. **6**(3), 511–520 (2010)

11. Chen, B., Huang, N.J.: Continuity of the solution mapping to parametric generalized vector equilibrium problems. J. Glob. Optim. **56**(4), 1515–1528 (2013)
12. Chen, C.R., Li, S.L., Teo, T.L.: Solution semicontinuity of parametric generalized vector equilibrium problems. J. Glob. Optim. **45**(2), 309–318 (2009)
13. Li, S.J., Li, X.B., Wang, L.N., Teo, K.L.: The Hölder continuity of solutions to generalized vector equilibrium problems. Eur. J. Oper. Res. **199**(2), 334–338 (2009)
14. Anh, L.Q., Khanh, P.Q.: Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces. J. Glob. Optim. **37**(3), 449–465 (2007)
15. Anh, L.Q., Khanh, P.Q., Tam, T.N.: On Hölder continuity of approximate solutions to parametric equilibrium problems. Nonlinear Anal. **75**(4), 2293–2303 (2012)
16. Han, Y., Gong, X.H.: Semicontinuity of solution mappings to parametric generalized vector equilibrium problems. Numer. Func. Anal. Optim. **37**(11), 1420–1437 (2016)
17. Han, Y.: Lipschitz continuity of approximate solution mappings to parametric generalized vector equilibrium problems. J. Optim. Theory Appl. **178**(3), 763–793 (2018)
18. Li, X.B., Li, S.J.: Continuity of approximate solution mapping for parametric equilibrium problems. J. Glob. Optim. **51**(3), 541–548 (2011)
19. Li, X.B., Li, S.J., Chen, C.R.: Lipschitz continuity of an approximate solution mapping to equilibrium problems. Taiwan. J. Math. **16**(3), 1027–1040 (2012)
20. Peng, Z.Y., Zhao, Y., Yang, X.M.: Semicontinuity of approximate solution mappings to parametric set-valued weak vector equilibrium problems. Numer. Func. Anal. Optim. **36**(4), 481–500 (2015)
21. Peng, Z.Y., Yang, X.M., Teo, K.L.: On the Hölder continuity of approximate solution mappings to parametric weak generalized Ky Fan inequality. J. Ind. Manag. Optim. **11**(2), 549–562 (2015)
22. Sadequi, Salehi Paydar M: Lipschitz continuity of an approximate solution mapping for parametric set-valued vector equilibrium problems. Optimization **65**(5), 1003–1021 (2016)
23. Anh, L.Q., Khanh, P.Q., Tam, T.N.: Continuity of approximate solution maps of primal and dual vector equilibrium problems. Optim. Lett. **13**(1), 201–211 (2019)
24. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
25. Göpfert, A., Riahi, H., Tammer, C., Zalinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
26. Kuroiwa, D.: Convexity for set-valued maps. Appl. Math. Lett. **9**(2), 97–101 (1996)
27. Anh, L.Q., Khanh, P.Q.: Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems. J. Math. Anal. Appl. **294**(2), 699–711 (2004)
28. Hu, S. and Papageorgiou, N.: Handbook of Multivalued Analysis, Volume I: Theory. Kluwer, Boston (1997)
29. Bao, T.Q., Gupta, P., Mordukhovich, B.S.: Necessary conditions in multiobjective optimization with equilibrium constraints. J. Optim. Theory Appl. **135**(2), 179–203 (2007)
30. Outrata, J.V.: A generalized mathematical program with equilibrium constraints. SIAM J. Control Optim. **38**(5), 1623–1638 (2000)
31. Mordukhovich, B.S.: Characterizations of linear suboptimality for mathematical programs with equilibrium constraints. Math. Program., Ser. B **120**(1), 261–283 (2009)
32. Lignola, M.B., Morgan, J.: Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution. J. Glob. Optim. **16**(1), 57–67 (2000)